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LINEAR COMPUTATIONAL STABILITY ANALYSIS FOR A MODIFIED SEMI-IMPLICIT INTEGRATION TECHNIQUE FOR GRAVITY OSCILLATIONS IN A TWO LAYER MODEL IN PHILLIPS σ COORDINATE

by

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1. In NMC Office Note 45, we presented an analysis of the linear computational stability of an explicit and an implicit formulation of the linear gravitational modes admitted by a two-layer model atmosphere expressed in Phillips' σ-coordinate. The implicit method was unconditionally stable, whereas the explicit method displayed only conditional stability. There were two critical phase speeds involved in the explicit method's conditional stability criterion. For an isothermal basic state, the larger phase speed was close to that of the "Lamb wave." The second, slower mode, was referred to as the "internal mode." The phase speed of the internal mode is commensurate with the speed of advective winds found in the atmosphere, whereas the "Lamb wave" phase speed is a good deal larger.

In practice, the implicit method requires the solution of a diagnostic boundary-value (elliptic) equation for each mode of oscillation which is treated implicitly. Thus, in the implicitly formulated two-layer model, one finds it necessary to solve two such boundary-value problems, at each time step. It is of some interest to examine the possibility of reducing the number of boundary value problems by utilizing a modified form of the semi-implicit method. It is the purpose of this note, to present the results of an analysis of the computational stability of such a modified scheme. The objective of the modified formulation is to treat the "Lamb wave" implicitly and the internal mode explicitly. The stability criterion, to be determined, was expected to relate only to the speed of the internal mode.

2. The linearized differential equations are:

$$u_t + \phi_x + \overline{\alpha}\,\sigma\, p^*_x = 0 \tag{1}$$

$$\phi_\sigma + \overline{\alpha}\, p^* + \alpha\, \overline{p^*} = 0 \tag{2}$$

$$p^*_t + \overline{p^*}\, u_x + \overline{p^*}\, \dot{\sigma}_\sigma = 0 \tag{3}$$

$$c_p\, T_t - \overline{\alpha}\,\sigma\, p^*_t + c_p\, \dot{\sigma}\, \overline{\Gamma} = 0 \tag{4}$$

$$\overline{\Gamma} = \overline{T}_\sigma - \overline{\alpha}\, \overline{p^*}/c_p \tag{4a}$$

$$\overline{p^*}\,\alpha\,\sigma + p^*\,\sigma\,\overline{\alpha} = RT \tag{5}$$

$$\sigma = p/p^* = \overline{p}/\overline{p^*} \tag{5a}$$

In order to apply the modified semi-implicit method, we first use eq. (3) in eq. (4) to get an alternative form of the thermodynamic equation:

$$c_p T_t + \bar{\alpha}\sigma [\, \overline{p^*} u_x + \overline{p^*} \dot{\sigma}_\sigma] + c_p \dot{\sigma} \bar{\Gamma} = 0 \tag{6}$$

Secondly, we note that the boundary conditions on $\dot{\sigma}$ at $\sigma = 0$ and $\sigma = 1$ are $\dot{\sigma} = 0$. This fact is used to replace (3) by two equations :

$$p^*_t + \overline{p^*} \int_0^1 u_x \, d\sigma = 0 \tag{7}$$

and

$$\overline{p^*} u_{\sigma x} + \overline{p^*} \dot{\sigma}_{\sigma\sigma} = 0 \tag{8}$$

3. The modified semi-implicit scheme is formalized by indicating the temporal discretization of the differential equations :

$$u^{n+1} - u^{n-1} + \Delta t[\, \phi_x^{n+1} + \phi_x^{n-1} + \bar{\alpha}\,\sigma(p_x^{*n+1} + p_x^{*n-1})] = 0 \tag{9}$$

$$\phi_\sigma^{n+1} + \phi_\sigma^{n-1} + \bar{\alpha}(p^{*n+1} + p^{*n-1}) + 2\, \overline{p^*}\, \alpha^n = 0 \tag{10}$$

$$p^{*n+1} - p^{*n-1} + \overline{p^*}\, \Delta t \int_0^1 (u_x^{n+1} + u_x^{n-1})\, d\sigma = 0 \tag{11}$$

$$\overline{p^*}\,(u^n_{\sigma x} + \dot{\sigma}^n_{\sigma\sigma}) = 0 \tag{12}$$

$$c_p(T^{n+1} - T^{n-1}) + 2\Delta t\, \bar{\alpha}\, \sigma[\, \overline{p^*} u_x^n + \overline{p^*} \dot{\sigma}_\sigma^n] + c_p \bar{\Gamma}\, 2\Delta t\, \dot{\sigma}^n = 0 \tag{13}$$

$$\overline{p^*}\, \alpha^n \sigma + p^{*n} \bar{\alpha}\, \sigma = RT^n \tag{14}$$

The basic idea formalized in eqs. (9-14) is the separation and different treatment of the "external" and "internal" modes of gravitational oscillation.

4. We shall now separate the time and horizontal variation from the vertical variation by writing for each dependent variable,

$$q = q(\sigma)\, e^{ikx}\, \zeta^n \tag{15}$$

On the right hand of (15), q is a function of σ alone. Stability requires that the solutions (15) exist with $|\zeta| \leq 1$. Upon substitution, we get

$$(\zeta^2 - 1)u + (ik\Delta t)(\zeta^2 + 1)[\phi + \overline{\alpha}\,\sigma\, p^*] = 0 \tag{16}$$

$$(\zeta^2 + 1)\phi_\sigma + \overline{\alpha}(\zeta^2 + 1)p^* + 2\,\overline{p^*}\,\zeta\,\alpha = 0 \tag{17}$$

$$(\zeta^2 - 1)p^* + (ik\Delta t)\overline{p^*}\,(\zeta^2 + 1)\int_0^1 u\, d\sigma = 0 \tag{18}$$

$$i\,k\,u_\sigma + \dot{\sigma}_{\sigma\sigma} = 0 \tag{19}$$

$$\begin{aligned} 0 = c_p(\zeta^2 - 1)T &+ 2(ik\Delta t)\overline{p^*}\,\overline{\alpha}\,\sigma\,\zeta\, u \\ &+ (2\Delta t)\zeta[\overline{p^*}\,\overline{\alpha}\,\dot{\sigma}_\sigma + c_p\overline{\Gamma}\,\dot{\sigma}] \end{aligned} \tag{20}$$

$$\overline{p^*}\,\alpha\,\sigma + p^*\,\overline{\alpha}\,\sigma = RT \tag{21}$$

5. Specialization to a two-layer model is now made,

$\sigma = 0$	———— $\dot{\sigma} = 0$ ————	ϕ_2
$\sigma = \frac{1}{4}$	- - - - $u_2,\ \alpha_2,\ T_2$ - - - -	$\overline{p}_2 = \frac{1}{4}\,\overline{p^*}$
$\sigma = \frac{1}{2}$	———— $\dot{\sigma}$ ————	ϕ_1
$\sigma = \frac{3}{4}$	- - - - $u_1,\ \alpha_1,\ T_1$ - - - -	$\overline{p}_1 = \frac{3}{4}\,\overline{p^*}$
$\sigma = 1$	———— $\dot{\sigma} = 0$ ————	$\phi = 0$

By introducing finite difference approximations to model the vertical variations, equations (16) through (21) are put into the following form.

$$(\zeta^2-1)u_1 + (\zeta^2+1)\ ik\Delta t\ \tfrac{1}{2}\ \phi_1 + (\zeta^2+1)\ ik\Delta t\ R\overline{T}_1\ r = 0$$
$$(\zeta^2-1)u_2 + (\zeta^2+1)\ ik\Delta t\ \tfrac{1}{2}(\phi_1+\phi_2) + (\zeta^2+1)\ ik\Delta t\ R\overline{T}_2\ r = 0 \qquad (22)$$

$$-(\zeta^2+1)\phi_1 + \tfrac{2}{3}(\zeta^2+1)R\overline{T}_1\ r + \overline{p^*}\ \zeta\ \alpha_1 = 0$$
$$-(\zeta^2+1)\phi_2 + (\zeta^2+1)\phi_1 + 2(\zeta^2+1)\ R\overline{T}_2\ r + \overline{p^*}\ \zeta\ \alpha_2 = 0 \qquad (23)$$

$$(\zeta^2-1)r + (\zeta^2+1)\ ik\Delta t\ \tfrac{1}{2}\ (u_1+u_2) = 0$$
$$ik(u_1-u_2) - 4\ \dot{\sigma} = 0 \qquad (24)$$

$$(\zeta^2-1)\ c_p\ T_1 + \zeta\ 2\ ik\Delta t\ R\overline{T}_1\ u_1 - \zeta\Delta t[4R\overline{T}_1-c_p\overline{\Gamma}]\ \dot{\sigma} = 0$$
$$(\zeta^2-1)\ c_p\ T_2 + \zeta\ 2\ ik\Delta t\ R\overline{T}_2\ u_2 + \zeta\Delta t[4R\overline{T}_2+c_p\overline{\Gamma}]\ \dot{\sigma} = 0 \qquad (25)$$

$$\overline{p}_1\ \alpha_1 + R\overline{T}_1\ r = RT_1$$
$$\overline{p}_2\ \alpha_2 + R\overline{T}_2\ r = RT_2 \qquad (26)$$

The new variable, r, is defined to be

$$r = p^*/\overline{p^*} \qquad (27a)$$

and

$$\overline{\Gamma} = [\ 2(\overline{T}_1-\overline{T}_2) - \tfrac{1}{2}(\overline{\alpha}_1+\overline{\alpha}_2)\ \overline{p^*}/c_p] \qquad (27b)$$

The system of ten equations in ten unknowns forms a homogeneous, linear set of simultaneous equations. The existence of a non-trivial solution is dependent upon the matrix of the coefficients having a zero valued determinant.

Provided that $\zeta \neq 0$, one may reduce the set of equations, by elimination of T_1, T_2 and α_1, α_2, to the following six equations :

$$2(\zeta^2-1)\ u_1 + (\zeta^2+1)\ ik\Delta t\ \phi_1 + 2(\zeta^2+1)\ ik\Delta t\ R\overline{T}_1\ r = 0$$
$$2(\zeta^2-1)\ u_2 + (\zeta^2+1)\ ik\Delta t\ (\phi_1+\phi_2) + 2(\zeta^2+1)\ ik\Delta t\ R\overline{T}_2\ r = 0 \qquad (28)$$

$$2(\zeta^2-1)\ r + (\zeta^2+1)\ ik\Delta t\ (u_1+u_2) = 0$$

$$4\ \dot{\sigma} - ik(u_1-u_2) = 0 \tag{29}$$

$$\zeta^2\ 8\ ik\Delta t\ c_1^2\ u_1 + 3(\zeta^2-1)(\zeta^2+1)\phi_1 - 2(\zeta-1)^2(\zeta^2-1)\ R\overline{T}_1\ r$$
$$+ 4\ \zeta^2\Delta t(R\overline{\Gamma}-4c_1^2)\dot{\sigma} = 0$$
$$\zeta^2\ 8\ ik\Delta t\ c_2^2\ u_2 + (\zeta^2-1)(\zeta^2+1)(\phi_2-\phi_1) - 2(\zeta-1)^2(\zeta^2-1)\ R\overline{T}_2\ r \tag{30}$$
$$+ 4\ \zeta^2(R\overline{\Gamma}+4c_2^2)\dot{\sigma} = 0$$

We have introduced the parameters

$$\begin{aligned} c_1^2 &\equiv \kappa\ R\overline{T}_1 \\ c_2^2 &\equiv \kappa\ R\overline{T}_2 \\ \kappa &= R/c_p \end{aligned} \tag{31}$$

The set of equations, (28), (29) and (30), may be put into matrix form,

$$L\ v = 0 \tag{32}$$

L is the matrix,

$$\begin{bmatrix} A & 0 & iB & 0 & iE_1 & 0 \\ 0 & A & iB & iB & iE_2 & 0 \\ iB & iB & 0 & 0 & A & 0 \\ -ik & ik & 0 & 0 & 0 & 4 \\ iC_1 & 0 & 3D & 0 & F_1 & 4G_1 \\ 0 & iC_2 & -D & D & F_2 & 4G_2 \end{bmatrix} \tag{33}$$

and v is the vector

$$\begin{bmatrix} u_1 \\ u_2 \\ \phi_1 \\ \phi_2 \\ r \\ \mathring{\sigma} \end{bmatrix} \tag{34}$$

The symbols used in (33) are defined by

$$\begin{aligned}
A &= 2(\zeta^2-1) & B &= \varepsilon(\zeta^2+1) \\
\varepsilon &= k\Delta t & D &= (\zeta^2-1)(\zeta^2+1) \\
C_1 &= 8\ \varepsilon\ c_1^2\ \zeta^2 & C_2 &= 8\ \varepsilon\ c_2^2\ \zeta^2 \\
E_1 &= 2\ \varepsilon\ R\overline{T}_1(\zeta^2+1) & E_2 &= 2\ \varepsilon\ R\overline{T}_2(\zeta^2+1) \\
F_1 &= -2\ R\overline{T}_1\ (\zeta-1)^2\ (\zeta^2-1) & F_2 &= -2\ R\overline{T}_2\ (\zeta-1)^2\ (\zeta^2-1) \\
G_1 &= k^{-1}\varepsilon(R\overline{\Gamma}-4c_1^2)\zeta^2 & G_2 &= k^{-1}\ \varepsilon\ (R\overline{\Gamma}+4\ c_2^2)\zeta^2
\end{aligned} \tag{35}$$

The frequency equation, obtained by requiring the determinant of L to vanish, has the form,

$$\begin{aligned}
&16(\zeta^2-1)(\zeta^2+1)^2\ \{\varepsilon^4[R\overline{\Gamma}(R\overline{T}_2-5R\overline{T}_1)(\zeta^2)(\zeta^2+1)^2 \\
&\quad + R\overline{\Gamma}(R\overline{T}_2-R\overline{T}_1)(\zeta^2)(\zeta^2+1)(\zeta-1)^2 + 4R\overline{\Gamma}(c_2^2-c_1^2)(\zeta^4)] \\
&\quad + \varepsilon^2[6(R\overline{T}_1+R\overline{T}_2)(\zeta^2-1)^2(\zeta^2+1)(\zeta^2-\zeta+1) \\
&\qquad + 12(c_1^2+c_2^2)(\zeta^2)(\zeta^2-1)^2 - 4R\overline{\Gamma}(\zeta^2)(\zeta^2-1)^2] \\
&\qquad + 6[(\zeta^2-1)^4]\} = 0
\end{aligned} \tag{36}$$

It will be noted that the factors, (ζ^2-1) and $(\zeta^2+1)^2$ are irrelevant to our analysis. The equation (36) is simplified by neglecting them. The expression, within the braces, is an eighth degree polynomial in ζ with real-valued coefficients. The eight roots will occur in conjugate pairs. There are basically two modes and, consequently, four physically relevant phase speeds. Our use of centered differences to approximate the time derivatives gives rise to an additional set of four phase speeds - the so-called computational modes. Each of these phase speeds is associated with one of the eight roots of equation (36).

6. The evaluation of the roots of the frequency equation (36) was made for an isothermal atmosphere at a temperature of 250°K. The value of $\overline{p^*}$ was set at 1000 mb.

As indicated in the introduction, it was anticipated that a conditional stability criterion would exist of the form,

$$k\Delta t \ c_* \equiv \varepsilon \ c_* \leqq 1 \ . \tag{37}$$

From previous analysis (NMC Office Note No. 45*), it was expected that c_* would have a value of approximately 82 m sec^{-1} when the isothermal basic state was employed.

The parameter ε in eq. (36) was allowed to have a set of values denoted by an integer index, m :

$$\varepsilon_m = \frac{2 \ \pi \ m}{3.81} \cdot 10^{-5} \ (\text{cm}^{-1} \ \text{sec}) \ . \tag{38}$$

For each value of ε_m $(m = 1,2,4,6,8,10)$, we evaluated the left-hand side of the simplified form of eq. (36) over the complex ζ-plane, including all of the unit circle.

For each such evaluation, we roughly approximated the loci of the zeroes, or roots, of the polynomial. It was anticipated that the roots would all lie on the unit circle, until the criterion (37) was violated. If our estimate of c_* was correct, the limiting value should have occurred when $m = 8.6$.

The results of our calculation are shown in figure 1. Only one half of the zeros are indicated, the other half were the complex conjugates of those shown.

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* The internal mode phase speed quoted is based upon $z_1 = 2.95$, a correction of the value given in the reference.

These results were not anticipated. The calculations have been carefully checked and appear to be correct.

The most significant point is the computational stability of the calculation for $m > 2$. If the usual explicit stability criterion were to apply to the fastest mode, $c_1 \approx 310$ m sec^{-1}, the limiting value of m would have been 2.

The first amplifying mode occurs when $m > 4$. Thus, the integration method may be stated to admit a time step, Δt, about two times larger than that of an explicit method.

The results suggest, however, that the first instability is associated with the fastest mode (Lamb wave), in spite of our attempt to treat it implicitly.

Another disturbing aspect of the results is the striking lack of symmetry about the imaginary axis. A comparison of the numerical phase angles with those expected from analytic calculation indicates that the physical-internal mode is estimated quite well, as is the computational-"Lamb" mode. Both the physical-"Lamb" mode and the computational-internal mode are underestimated. The only explanation which we can offer for this behavior is the "mixed character" of the approximations used in the two forms of the continuity equation and in the thermodynamic equation.

Although our results are ambiguous with respect to the merits of the proposed modification of the implicit scheme, it is concluded that further investigation of the method appears warranted.

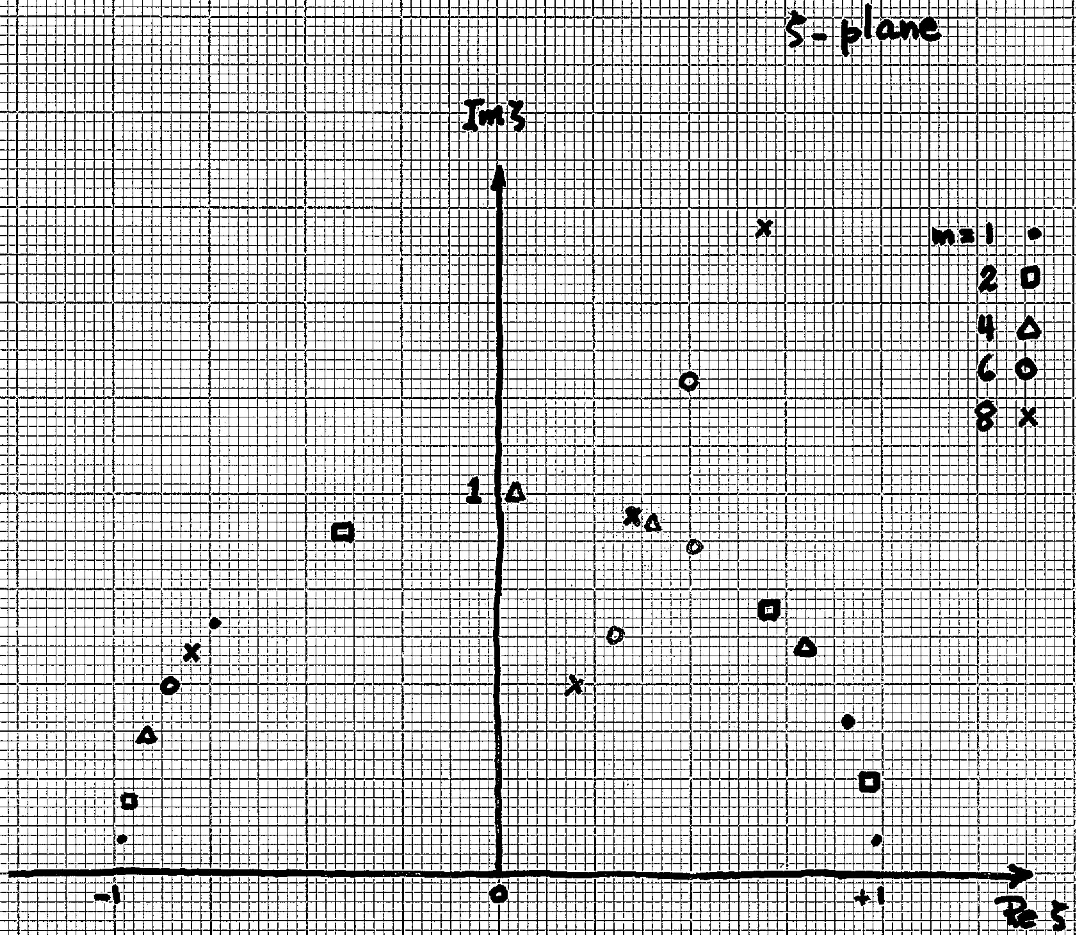

Figure 1. Loci of the zeros of the polynomial in ζ for an isothermal atmosphere, $\bar{T} = 250°K$, $p^* = 1000$ mb, as a function of $\varepsilon = k\Delta t$; $\varepsilon_m = 2\pi m/3.81 \times 10^{-5}$ ($cm^{-1}sec^{+1}$).